Multifractality and electron-electron interaction at Anderson transitions

Igor Burmistrov

in collaboration with

Igor Gornyi (KIT, Ioffe Inst., Landau Inst.)

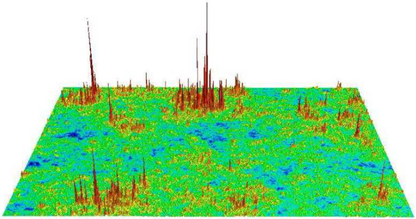
Alexander Mirlin (KIT, Peterburg Nucl. Phys. Inst., Landau Inst.)

PRL 111, 066601 (2013) & PRB 89, 035430 (2014) & PRB 91, 085427 (2015)

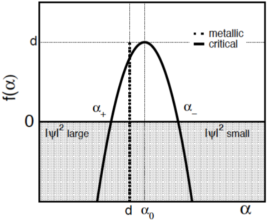
[Wegner (1980,1987); Kravtsov, Lerner (1985); Pruisken(1985); Castellani, Peliti (1986)]

$$\int_{|\boldsymbol{r}|\leqslant L} d^d \boldsymbol{r} \left\langle \left|\psi_E(\boldsymbol{r})\right|^{2q} \right\rangle_{\text{dis}} \sim L^{-\tau_q}, \quad \tau_q = \begin{cases} d(q-1), & \text{metal}, \\ d(q-1)+\Delta_q, & \text{criticality}, \\ 0, & \text{insulator} \end{cases}$$

- multifractal exponent $\Delta_q \leqslant 0$ is nonlinear function of q
- Legendre transform of τ_q: $f(\alpha) = q\alpha - \tau_q, \quad \alpha = d\tau_q/dq$
- $L^{f(\alpha)}$ measures a set of points where $\left|\psi_E\right|^2 \sim L^{-\alpha}$



[adapted from Evers, Mildenberger, Mirlin]



[adapted from Evers, Mirlin (2008)]

- local density of states (LDOS) in the cube of size L

$$\rho(E, \boldsymbol{r}) = \sum_{\alpha} \left|\psi_\alpha(\boldsymbol{r})\right|^2 \delta(E - \epsilon_\alpha)$$

where $\psi_\alpha(\boldsymbol{r})$ and ϵ_α w. f. and energy for a given disorder

- scaling of the moments of LDOS

$$\left\langle [\rho(E, \boldsymbol{r})]^q \right\rangle_{\text{dis}} \sim L^{-\Delta_q}, \qquad q = 0, 1, 2, \dots$$

- strong mesoscopic fluct. of LDOS due to multifractality: $\Delta_q \leqslant 0$

- spatial correlations of LDOS

$$\langle \rho(E, \boldsymbol{r}) \rho(E, \boldsymbol{r} + \boldsymbol{R}) \rangle_{\text{dis}} \sim (R/L)^{\Delta_2}, \qquad R \ll L$$

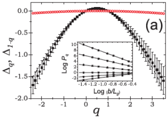
examples for Anderson transitions in $d = 2$:

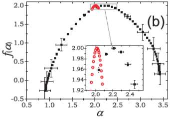
$\Delta_2 = -0.34$ (class AII, spin-orbit coupling, numerics)
$\Delta_2 = -0.52$ (class A, integer qHe, numerics)
$\Delta_2 = -1/4$ (class C, spin qHe, exact)

[see for a review, Evers&Mirlin (2008)]

- ultrasound speckle in the system of randomly packed Al beads

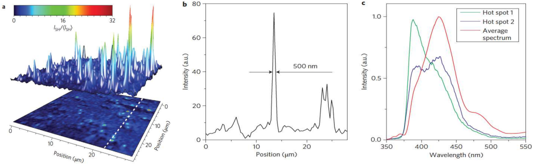
[Faez et al., 2009]





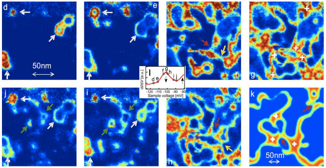
- localization of light in an array of dielectric nano-needles

[Mascheck et al., 2012]



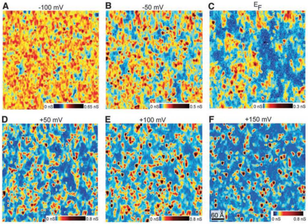
- differential conductance in 2DEG in InSb at $B = 12$ T

[Morgenstern et al. (2012)]



- differential conductance over an area of 500 Å× 500 Å in $Ga_{1-x}Mn_xAs$ with $x = 1.5\%$

[Richardella et al. (2010)]



[Altshuler, Aronov, Lee (1980), Finkelstein (1983), Castellani, DiCastro, Lee, Ma (1984)]
[Nazarov (1989), Levitov, Shytov (1997), Kamenev, Andreev (1999)]

- Suppression of LDOS at the Fermi energy in $d = 2$ $(L = \infty)$

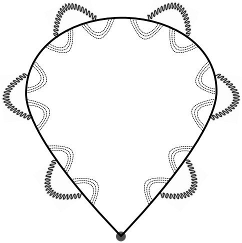
$$\langle \rho(E, \mathbf{r}) \rangle_{\text{dis}} \sim \exp\left(-\frac{1}{4\pi g} \ln(|E|\tau) \ln \frac{|E|}{D^2 \kappa^4 \tau}\right)$$

where

g - conductance in units e^2/h,
D - diffusion coefficient,
$\kappa = e^2 \rho_0/\varepsilon$ - inverse static screening length

- Zero bias anomaly in $d = 2 + \epsilon$ at Anderson transition criticality

$$\langle \rho(E, \mathbf{r}) \rangle_{\text{dis}} \sim |E|^{\beta}, \qquad \beta = O(1)$$



in the absence of interaction average LDOS is non-critical ($\beta = 0$) for Wigner-Dyson classes
[see for a review, Finkelstein (1990), Kirkpatrick&Belitz (1994)]

Question to answer

How Coulomb interaction affects mesoscopic fluctuations of the local density of states?

- free electrons in d dimensions

$$H_0 = \int d^d\boldsymbol{r}\,\overline{\psi}(\boldsymbol{r})\epsilon(\nabla)\psi(\boldsymbol{r}),$$

$\epsilon(\boldsymbol{p}) = \boldsymbol{p}^2/2m$ – class AI (spin-rotational and time-reversed symmetries are preserved)
$\epsilon(\boldsymbol{p}) = -iv\boldsymbol{p}\times\boldsymbol{\sigma}$ – class AII (time-reversed symmetry are preserved)
$\epsilon(\boldsymbol{p}) = (\boldsymbol{p} - e\boldsymbol{A})^2/2m$ – class A (spin-rotational symmetry is preserved)

- scattering off white-noise random potential

$$H_{\rm dis} = \int d^d\boldsymbol{r}\,\overline{\psi}(\boldsymbol{r})V(\boldsymbol{r})\psi(\boldsymbol{r}), \qquad \langle V(\boldsymbol{r})V(0)\rangle = \frac{1}{2\pi\rho_0\tau}\delta(\boldsymbol{r})$$

- Coulomb interaction:

$$H_{\rm int} = \frac{1}{2}\int d^d\boldsymbol{r}_1 d^d\boldsymbol{r}_2\,\frac{e^2}{\varepsilon|\boldsymbol{r}_1 - \boldsymbol{r}_2|}\,\overline{\psi}_\sigma(\boldsymbol{r}_1)\psi_\sigma(\boldsymbol{r}_1)\overline{\psi}_{\sigma'}(\boldsymbol{r}_2)\psi_{\sigma'}(\boldsymbol{r}_2)$$

- disorder-averaged moments of the LDOS

$$\Big\langle[\rho(E,\boldsymbol{r})]^q\Big\rangle_{\rm dis} = \Big\langle\Big[-\frac{1}{\pi}\,{\rm Im}\,\mathcal{G}^R(E,\boldsymbol{r},\boldsymbol{r})\Big]^q\Big\rangle_{\rm dis}, \qquad \mathcal{G}^R(\boldsymbol{r}t;\boldsymbol{r}'t') = -i\theta(t-t')\Big\langle\{\overline{\psi}(\boldsymbol{r}t),\psi(\boldsymbol{r}'t')\}\Big\rangle$$

- assumptions (diffusive regime) $\mu \gg \frac{1}{\tau} \gg T, |E|$, where μ – chemical potential, τ – mean-free time for scattering off potential impurities, T – temperature E – energy measured from the chemical potential.

[Finkelstein(1983)]

- action for the nonlinear sigma-model

$$\mathcal{S}[Q] = -\frac{g}{32}\int d\boldsymbol{r}\,\mathrm{tr}(\nabla Q)^2 + 4\pi T Z_\omega \int d\boldsymbol{r}\,\mathrm{tr}\,\eta Q - \frac{\pi T}{4}\sum_{r=0,3}\sum_{j=0,\dots,3}\sum_{\alpha,n}\int d\boldsymbol{r}\,\Gamma_j\,\mathrm{tr}\,I^\alpha_n t_{rj} Q\,\mathrm{tr}\,I^\alpha_{-n} t_{rj} Q$$
$$-\frac{\pi T}{4}\sum_{r=1,2}\sum_{\alpha,n}\int d\boldsymbol{r}\,\Gamma_c\,\mathrm{tr}\,L^\alpha_n t_{r0} Q\,\mathrm{tr}\,L^\alpha_n t_{r0} Q$$

where the matrix field Q (in Matsubara, particle-hole, spin and replica spaces) satisfies

$$Q^2(\boldsymbol{r}) = 1, \qquad \mathrm{tr}\,Q(\boldsymbol{r}) = 0, \qquad Q^\dagger(\boldsymbol{r}) = C^T Q(\boldsymbol{r}) C,$$

g – conductivity in units e^2/h, $\Gamma_0 \equiv \Gamma_s$ – interaction amplitude in the singlet channel, $\Gamma_1, \Gamma_2, \Gamma_3 \equiv \Gamma_t$ – interaction amplitude in the triplet channel, Γ_c – Cooper channel interaction amplitude, $\tau_{rj} = \tau_r \otimes s_j$, $\boldsymbol{\tau}$, $\boldsymbol{s}$ are Pauli matrices in particle-hole and spin spaces, $C = it_{12}$, and matrices

$$\Lambda^{\alpha\beta}_{nm} = \mathrm{sgn}\,n\,\delta_{nm}\delta^{\alpha\beta}, \qquad \eta^{\alpha\beta}_{nm} = n\,\delta_{nm}\delta^{\alpha\beta}, \qquad (I^\gamma_k)^{\alpha\beta}_{nm} = \delta_{n-m,k}\delta^{\alpha\beta}\delta^{\alpha\gamma}, \qquad (L^\gamma_k)^{\alpha\beta}_{nm} = \delta_{n+m,k}\delta^{\alpha\beta}\delta^{\alpha\gamma}$$

- disorder-averaged LDOS

$$\left\langle \rho(E,\boldsymbol{r}) \right\rangle_{\text{dis}} = \frac{\rho_0}{4} \langle K_1(E) \rangle_{\mathcal{S}},\ K_1(E) = \text{Re}\, P_1^R(E),\ P_1(i\varepsilon_n) = \text{sp}\, Q_{nn}^{\alpha\alpha}(\boldsymbol{r})$$

where $\varepsilon_n = \pi T(2n+1)$ is fermionic Matsubara frequencies

- disorder-averaged 2d moment of LDOS

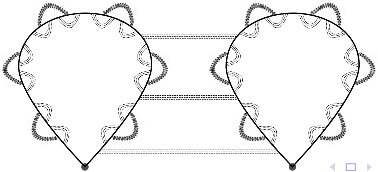
$$\left\langle \rho(E,\boldsymbol{r})\rho(E',\boldsymbol{r}) \right\rangle_{\text{dis}} = (\rho_0^2/32) \langle K_2(E,E') \rangle_{\mathcal{S}}$$

$$K_2(E,E') = \text{Re}\left[P_2^{RR}(E,E') - P_2^{RA}(E,E')\right]$$

$$P_2(i\varepsilon_n, i\varepsilon_m) = \text{sp}\, Q_{nn}^{\alpha_1\alpha_1}(\boldsymbol{r})\, \text{sp}\, Q_{mm}^{\alpha_2\alpha_2}(\boldsymbol{r}) - 2\, \text{sp}\, Q_{nm}^{\alpha_1\alpha_2}(\boldsymbol{r}) Q_{mn}^{\alpha_2\alpha_1}(\boldsymbol{r})$$

...

- K_q are eigenoperators of renormalization group transformations. Renormalization of K_q by means of perturbation theory around $Q = \Lambda$.



- scaling of the moments of LDOS ($L_E \sim |E|^{-1/z}$, $\xi \sim |t - t_*|^{-\nu}$, $\theta = \boldsymbol{\beta} z$)

$$\left\langle [\rho(E, \mathbf{r})]^q \right\rangle \sim \langle \rho(E) \rangle^q \mathcal{L}^{-\Delta_q} \sim \mathcal{L}^{-\theta q - \Delta_q}, \qquad \mathcal{L} = \min\{L_E, L, \xi\},$$

where multifractal exponents Δ_q are determined by anomalous dimensions

$$\zeta_q(t, \gamma_j) = \frac{q(1-q)}{2} \Big[2t + \big(c(\gamma_s) + 3c(\gamma_t) - 2\gamma_c \big) t^2 \Big] + O(t^3),$$

$$c(\gamma) = 2 + \frac{2+\gamma}{\gamma} \mathrm{li}_2(-\gamma) + \frac{1+\gamma}{2\gamma} \ln^2(1+\gamma).$$

Note that $\gamma = \Gamma / Z_\omega$. For the Coloumb interaction, $\gamma_s = -1$. The function $c(\gamma)$ monotonously decease between $c(-1) = 2 - \pi^2/6$ and $c(0) = 0$.

average LDOS exponent $\theta = \zeta^*$ where

$$\zeta(t, \gamma_j) = -\Big[\ln(1+\gamma_s) + 3\ln(1+\gamma_t) + 2\gamma_c \Big] \frac{t}{2} + O(t^2)$$

[Finkelstein (1983), Castellani, DiCastro, Lee, Ma (1984)]

cf. anomalous dimensions for non-interacting electrons

$$\zeta_q^{(\mathrm{n})}(t) = q(1-q)t + O(t^4).$$

[Höf&Wegner (1986), Wegner (1987)]

[Hikami(1983), Castellani, DiCastro, Lee, Ma(1984), Finkelstein(1984), Bernreuther, Wegner(1986)]
[Baranov, Pruisken, Škorić (1999), Baranov, Burmistrov, Pruisken (2002)]

- no interaction

$$\beta(t) = \epsilon t - (t/2)^3 - \frac{3}{4}(t/2)^5 + O(t^6)$$

$$t_*^{(\mathrm{n})} = 2\sqrt{2\epsilon}\left(1 - \frac{3\epsilon}{4}\right) + O(\epsilon^{5/2})$$

$$\nu_{\mathrm{n}} = 1/(2\epsilon) - 3/4 + O(\epsilon)$$

$$z_{\mathrm{n}} = d = 2 + \epsilon$$

$$\boldsymbol{\beta}_{\mathrm{n}} = 0$$

$$\Delta_q^{(\mathrm{n})} = q(1-q)\left(\frac{\epsilon}{2}\right)^{1/2} - \frac{3\zeta(3)}{32} q^2(q-1)^2\epsilon^2 + O(\epsilon^{5/2})$$

- Coulomb interaction

$$\beta(t) = \epsilon t - t^2 - A t^3 + O(t^4)$$

$$t_* = \epsilon(1 - A\epsilon) + O(\epsilon^3)$$

$$\nu = 1/\epsilon - A + O(\epsilon)$$

$$z = 2 + \epsilon/2 + B\epsilon^2 + O(\epsilon^3)$$

$$\boldsymbol{\beta} = 1/2 + O(\epsilon)$$

$$\Delta_q = \frac{q(1-q)\epsilon}{4} + \frac{q(1-q)\epsilon^2}{4}\left(1 - A - \frac{\pi^2}{12}\right) + O(\epsilon^3)$$

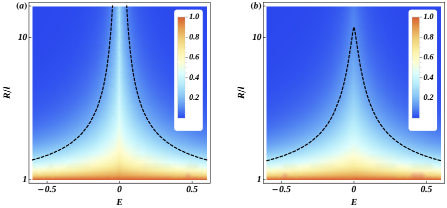
multifractality is supressed by Coulomb interaction in LDOS, $\theta q + \Delta_q > 0$, but only weakens in normalized LDOS, $|\Delta_q| < |\Delta_q|^{(\mathrm{n})}$ (for not too large values of q)

$$A = \frac{139}{96} + \frac{(\pi^2-18)^2}{192} + \frac{19}{32}\zeta(3) + \left(1 + \frac{\pi^2}{48}\right)\ln^2 2 - \left(44 - \frac{\pi^2}{2} + 7\zeta(3)\right)\frac{\ln 2}{16} + \mathcal{G} - \frac{1}{48}\ln^4 2 - \frac{1}{2}\mathrm{li}_4\left(\frac{1}{2}\right) \approx 1.64, \qquad B = \frac{1}{4}\left(2A - \frac{\pi^2}{6} - 3\right) \approx -0.34$$

$$\frac{\langle\langle \rho(E,\mathbf{r})\rho(E,\mathbf{r}+\boldsymbol{R})\rangle\rangle}{\langle\langle \rho^2(E,\mathbf{r})\rangle\rangle} \sim \begin{cases} (R/\mathcal{L})^{\Delta_2}, & R \ll \mathcal{L} = \min\{|E|^{-1/z}, (t_* - t)^{-\nu}\} \\ 0, & \mathcal{L} \ll R \end{cases}$$

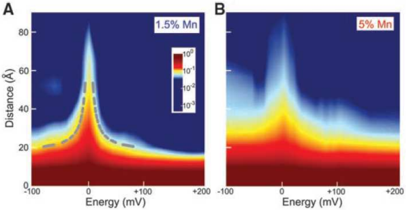
critical phase

$t = t_*$



metallic phase

$t < t_*$

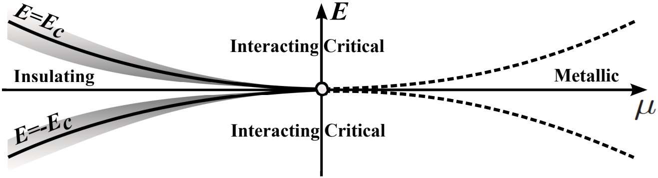


[Richardella et al. (2010)]

Note: characteristic energy scale $E_c \sim |t_* - t|^{\nu z}$

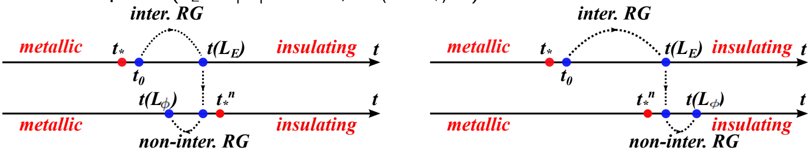
- interaction favors localization, $t_* < t_*^{(n)}$ (e.g. $d = 2 + \epsilon$)

- phase diagram near interacting critical point $t = t_*$ ($\mu = \mu_*$):



The insulating phase $t > t_*$ /class A, Coulomb interaction

- two-step RG ($L_E \sim |E|^{-1/z} \ll \xi \sim (t - t_*)^{-\nu}$):



$t(L_E) < t_*^{(\mathrm{n})}$ $\qquad\qquad$ $t(L_E) > t_*^{(\mathrm{n})}$

- mobility edge E_c for single particle excitations

$$t(L_{E_c}) = t_*^{(\mathrm{n})} \qquad \Longrightarrow \qquad E_c \sim (t - t_*)^{\nu z} \sim (\mu_* - \mu)^{\nu z}$$

- divergent localization and dephasing lengths

$$\xi(E) \sim \xi \big| |E|/E_c - 1 \big|^{-\nu_{\mathrm{n}}}, \qquad L_\phi(E) \sim L_E \begin{cases} \left(|E|/E_c - 1\right)^{-z_{\mathrm{n}}^\phi}, & |E| - E_c \ll E_c \\ \infty, & |E| < E_c \end{cases}, \qquad z_{\mathrm{n}}^\phi = \frac{d^2}{d + \Delta_2^{(\mathrm{n})}}$$

N.B.: the conditions $z\nu > 1$ and $z_{\mathrm{n}}^\phi \nu_{\mathrm{n}} > 1$ hold

- interacting criticality $|E| \gg E_c$ ($L_E \sim |E|^{-1/z} \ll \xi \sim (t-t_*)^{-\nu}$):

$$\frac{\langle \rho^q(E)\rangle}{\langle \rho(E)\rangle^q} \sim L_E^{-\Delta_q}$$

- noninteracting criticality above the mobility edge $|E| - E_c \ll E_c$:

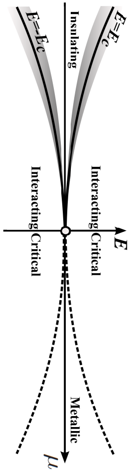
$$\frac{\langle \rho^q(E)\rangle}{\langle \rho(E)\rangle^q} \sim \xi^{-\Delta_q} \begin{cases} \left(\frac{L_\phi(E)}{\xi}\right)^{-\Delta_2^{(n)}}, & L_\phi(E) \ll L \\ \left(\frac{L}{\xi}\right)^{-\Delta_2^{(n)}}, & L_\phi(E) \gg L \end{cases}$$

- noninteracting criticality below the mobility edge $E_c - |E| \ll E_c$:

$$\frac{\langle \rho^q(E)\rangle}{\langle \rho(E)\rangle^q} \sim \xi^{-\Delta_q} \left(\frac{L}{\xi(E)}\right)^{d(q-1)} \begin{cases} \left(\frac{\xi(E)}{\xi}\right)^{-\Delta_2^{(n)}}, & \xi(E) \ll L \\ \left(\frac{L}{\xi}\right)^{-\Delta_2^{(n)}}, & \xi(E) \gg L \end{cases}$$

- deep below the mobility edge $|E| \ll E_c$ ($L_E \gg \xi$)

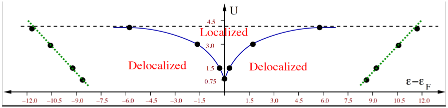
$$\frac{\langle \rho^q(E)\rangle}{\langle \rho(E)\rangle^q} \sim \xi^{-\Delta_q} \left(\frac{L}{\xi}\right)^{d(q-1)}$$

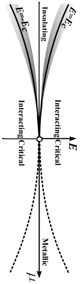


[Amini, Kravtsov, Müller (2013)]

- LDOS correlation function made from Hartree-Fock w.f.

$$\frac{\langle \rho_{HF}(E,\boldsymbol{r})\rho_{HF}(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho_{HF}^2(E,\boldsymbol{r})\rangle^2}$$





for a short-ranged interaction the two-particle mobility edge

$$E_{c2} \sim (\mu_c - \mu)^{d\nu_n}$$

[Imry (1995); Jacquod, Shepelyansky (1997); Shepelyansky (2000)]

for Coulomb interaction we expect that

$$E_{c2} \sim E_c \sim (\mu_c - \mu)^{z\nu}$$

neutral e-h excitations are delocalized due to dipole-dipole $(1/R^3)$ interaction in $d \geqslant d_c$

[$d_c = 3$: Fleishman, Anderson (1980); Levitov (1990)]

[$d_c = 3/2$: Burin (2006); Yao et al. (2013)]

we expect that

(i) for $d \geqslant d_c$ single-particle excitations will be localized despite emitting of neutral e-h excitations

(ii) dephasing length near and below E_c will be modified by neutral e-h excitations

(iii) LDOS correlations will be smeared by neutral e-h excitations below E_c: $L \to L_\phi^{eh}$

Conclusions

- multifractality in LDOS does exist in interacting disordered systems
- anomalous dimensions controlling multifractal exponents are computed within the two-loop approximation for different symmetry classes
- multifractal exponents are different from the non-interacting case
- on the insulating side of the Anderson transition there is the mobility edge for single-particle excitations
- scaling of LDOS near the mobility edge is controlled by non-interacting critical exponents
- our results provide qualitative understanding of experiments by Richardella et al. and numerics by Amini et al.

Exercises

1. Using the relation
$$\left\langle [\rho(E,\boldsymbol{r})]^2 \right\rangle_{\text{dis}} \sim L^{-\Delta_2},$$
to demonstrate that
$$\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle_{\text{dis}} \sim (R/L)^{\Delta_2}, \qquad R \ll L.$$

2. Using the relation
$$\left\langle [\rho(E,\boldsymbol{r})]^2 \right\rangle_{\text{dis}} \sim L^{-\Delta_2},$$
to demonstrate that ($L_\omega \sim |\omega|^{-1/d}$)
$$\langle \rho(E,\boldsymbol{r})\rho(E+\omega,\boldsymbol{R})\rangle_{\text{dis}} \sim L_\omega^{-\Delta_2}, \qquad L_\omega \ll L$$

3.* There is the unitary matrix $Q^{pq}_{\alpha\beta}$ in replica $\alpha,\beta = 1,\ldots,n$ and R/A $p,q = \pm 1$ spaces. To compute the average of the following operator $Q^{pq}_{\alpha\beta}Q^{qp}_{\beta\alpha}$ ($\alpha \neq \beta$) over rotations $U^{kl}_{\mu\nu} = (1/2)(1+kl)\delta^{kl}U^{k}_{\mu\nu}$.